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Square-difference factor absorbing ideals of a commutative ring

David F. Anderson*, Ayman Badawi†,§ and Jim Coykendall‡

*Department of Mathematics, The University of Tennessee
Knoxville, TN 37996-1320, USA

†Department of Mathematics and Statistics
The American University of Sharjah
P.O. Box 26666, Sharjah, United Arab Emirates

‡School of Mathematical and Statistical Sciences, Clemson University
205 Long Hall, Clemson, SC 29634, USA
§abadawi@aus.edu

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Let R be a commutative ring with $1 \neq 0$. A proper ideal I of R is a *square-difference factor absorbing ideal* (sdf-absorbing ideal) of R if whenever $a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b \in I$ or $a - b \in I$. In this paper, we introduce and investigate sdf-absorbing ideals.

Keywords: Square-difference factor absorbing ideal; weakly square-difference factor absorbing ideal; prime ideal; weakly prime ideal; radical ideal; two-absorbing ideal; n-absorbing ideal.

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1. Introduction

In this paper, we introduce and study square-difference factor absorbing ideals of a commutative ring R with nonzero identity, where a proper ideal I of R is a *square-difference factor absorbing ideal* (sdf-absorbing ideal) of R if whenever $a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b \in I$ or $a - b \in I$. A prime ideal is an sdf-absorbing ideal (but not conversely). In particular, $\{0\}$ is an sdf-absorbing ideal in any integral domain. For other generalizations of prime ideals, see [2, 3, 6]. We also introduce and briefly study weakly square-difference factor absorbing ideals, where a proper ideal I of R is a *weakly square-difference factor absorbing ideal* (weakly sdf-absorbing ideal) of R if whenever $0 \neq a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b \in I$ or $a - b \in I$.

§Corresponding author.

In Sec. 2, we give some basic properties of sdf-absorbing ideals. We show that a nonzero sdf-absorbing ideal is a radical ideal (Theorem 2.2), and the converse holds when $\mathrm{char}(R) = 2$ (Theorem 2.4). Moreover, a nonzero sdf-absorbing ideal is a prime ideal when $2 \in U(R)$ (Theorem 2.6). Throughout this paper, properties of $2 \in R$ will play an important role. In Sec. 3, we study when all proper ideals or nonzero proper ideals of a commutative ring are sdf-absorbing ideals. In particular, every nonzero proper ideal of a commutative ring R is an sdf-absorbing ideal of R if and only if $R/\mathrm{nil}(R)$ is a von Neumann regular ring (Theorem 3.1). In addition, we determine when every proper ideal or nonzero proper ideal of a commutative von Neumann regular ring is an sdf-absorbing ideal (Theorems 3.5–3.7). In Sec. 4, we give several additional results about sdf-absorbing ideals. For example, we determine the sdf-absorbing ideals in a PID (Corollary 4.3) and the direct product of two commutative rings (Theorem 4.12). We also study sdf-absorbing ideals in polynomial rings (Theorems 4.5 and 4.8), idealizations (Theorem 4.16), amalgamation rings (Theorem 4.19), and $D + M$ constructions (Theorem 4.21). In Sec. 5, the final section, we briefly study weakly sdf-absorbing ideals. Several results for sdf-absorbing ideals have analogs for weakly sdf-absorbing ideals. Many examples are given throughout to illustrate the results.

We assume throughout that all rings are commutative with nonzero identity and that $f(1) = 1$ for all ring homomorphisms $f : R \to T$. Let R be a commutative ing. Then $\dim(R)$ denotes the Krull dimension of R, $\mathrm{char}(R)$ the characteristic of R, $J(R)$ the Jacobson radical of R, $\mathrm{nil}(R)$ the ideal of nilpotent elements of R, $Z(R)$ the set of zero-divisors of R, and $U(R)$ the group of units of R. The ring R is *reduced* if $\mathrm{nil}(R) = \{0\}$. Recall that a commutative ring R is *von Neumann regular* if for every $x \in R$, there is a $y \in R$ such that $x^2y = x$. Equivalently, R is von Neumann regular if and only if R is reduced and $\dim(R) = 0$ [11, Theorem 3.1].

As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, and $\mathbb{F}_q$ will denote the integers, rationals, integers modulo n, and the finite field with q elements, respectively. For any undefined concepts or terminology, see [10–12].

2. Properties of sdf-Absorbing Ideals

In this section, we give some basic properties of square-difference factor absorbing ideals. We begin with the definition.

Definition 2.1. A proper ideal I of a commutative ring R is a *square-difference factor absorbing ideal* (sdf-absorbing ideal) of R if whenever $a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b \in I$ or $a - b \in I$.

Clearly a prime ideal is an sdf-absorbing ideal. Although the converse may fail (see Example 2.8(a)), we next show that nonzero sdf-absorbing ideals are always radical ideals.

Theorem 2.2. *Let I be a nonzero sdf-absorbing ideal of a commutative ring R. Then I is a radical ideal of R.*

Proof. It is sufficient to show that for $0 \neq a \in R$, we have $a \in I$ whenever $a^2 \in I$. Since I is a nonzero ideal of R, there is a $0 \neq i \in I$. Thus $a^2 - i^2 \in I$; so $a + i \in I$ or $a - i \in I$ since I is an sdf-absorbing ideal of R. Hence, $a \in I$, and thus I is a radical ideal of R. □

Remark 2.3. (a) The "nonzero" hypothesis is needed in Theorem 2.2. It is easily verified that $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_4$ (cf. Theorem 4.11), but not a radical ideal of $\mathbb{Z}_4$.

(b) Theorem 2.2 also shows that the "$a, b \neq 0$" hypothesis is not needed in the definition of sdf-absorbing ideal when the ideal is nonzero.

A radical ideal need not be an sdf-absorbing ideal, see Example 2.8(a). However, the converse of Theorem 2.2 does hold when $\mathrm{char}(R) = 2$, and the "nonzero ideal" hypothesis is not needed.

Theorem 2.4. *Let I be a radical ideal of a commutative ring R with char$(R) = 2$. Then I is an sdf-absorbing ideal of R.*

Proof. Let I be a radical ideal of R and $a^2 - b^2 \in I$ for $0 \neq a, b \in R$. Since $\mathrm{char}(R) = 2$, we have $(a + b)^2 = a^2 + b^2 = a^2 - b^2 \in I$, and thus $a + b \in I$ since I is a radical ideal of R. Hence, I is an sdf-absorbing ideal of R. □

If $\mathrm{char}(R) = 2$, then $a - b = a + b$. The next result determines when $a^2 - b^2 \in I$ for I an sdf-absorbing ideal of R implies both $a + b, a - b \in I$.

Theorem 2.5. *Let I be an sdf-absorbing ideal of a commutative ring R. Then the following statements are equivalent*:

(a) *If $a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b, a - b \in I$.*
(b) $2 \in I$.
(c) $\mathrm{char}(R/I) = 2$.

Proof. (a) ⇒ (b) Let $a = b = 1$. Then $a^2 - b^2 = 0 \in I$, and thus $2 = 1+1 = a+b \in I$ by hypothesis.

(b) ⇒ (a) Assume that $a^2 - b^2 \in I$ for $0 \neq a, b \in R$. Then $a + b \in I$ or $a - b \in I$ since I is an sdf-absorbing ideal of R, and $2b \in I$ since $2 \in I$. Thus $a - b = (a + b) - 2b \in I$ if $a + b \in I$, and $a + b = (a - b) + 2b \in I$ if $a - b \in I$.

(b) ⇔ (c) This is clear. □

The next result gives a case where sdf-absorbing ideals are prime ideals. It is easily verified that $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_9$ (cf. Theorem 4.11); so the "nonzero" hypothesis is needed in the following theorem.

Theorem 2.6. *Let I be a nonzero sdf-absorbing ideal of a commutative ring R with $2 \in U(R)$. Then I is a prime ideal of R.*

Proof. Let I be a nonzero sdf-absorbing ideal of R and $xy \in I$ for $x, y \in R$. We may assume that $x, y \neq 0$. First, assume that $y \neq x$ and $y \neq -x$. Let $a = (x+y)/2$, $b = (x-y)/2 \in R$. Since $y \neq x$ and $y \neq -x$, we have $a^2 - b^2 = xy \in I$ and $a, b \neq 0$. Thus $x = a + b \in I$ or $y = a - b \in I$ since I is an sdf-absorbing ideal of R. Next, assume that $y = x$ or $y = -x$. Then $x^2 \in I$, and hence $x \in I$ since I is a radical ideal of R by Theorem 2.2. Thus I is a prime ideal of R. □

The following criterion for an ideal to be an sdf-absorbing ideal will often prove useful.

Theorem 2.7. *Let I be a proper ideal of a commutative ring R. Then the following statements are equivalent.*

(a) *I is an sdf-absorbing ideal of R.*
(b) *If $ab \in I$ for $a, b \in R \backslash I$, then the system of linear equations $X + Y = a$, $X - Y = b$ has no nonzero solution in R (i.e. there are no $0 \neq x, y \in R$ that satisfy both equations).*

Proof. (a) $\Rightarrow$ (b) Suppose that I is an sdf-absorbing ideal of R, $ab \in I$ for $a, b \in R\backslash I$, and the system of linear equations $X + Y = a$, $X - Y = b$ has a solution in R for some $0 \neq x, y \in R$. Then $x^2 - y^2 = ab \in I$, but $x + y = a \notin I$ and $x - y = b \notin I$, a contradiction.

(b) $\Rightarrow$ (a) Assume that $x^2 - y^2 \in I$ for $0 \neq x, y \in R$. Let $a = x + y$ and $b = x - y$. Then $ab = x^2 - y^2 \in I$, and the system of linear equations $X + Y = a$, $X - Y = b$ has a solution in R for $0 \neq x, y \in R$. Thus $x + y = a \in I$ or $x - y = b \in I$, and hence I is an sdf-absorbing ideal of R. □

We next give several examples of sdf-absorbing ideals.

Example 2.8. (a) Using Theorem 2.7, one can easily verify that a proper ideal I of $\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$ if and only if I is a prime ideal of $\mathbb{Z}$ or $I = 2q\mathbb{Z}$ for some odd prime integer q. Note that if p, q are nonassociate odd prime integers, then $pq\mathbb{Z}$ is a radical ideal of $\mathbb{Z}$ which is not an sdf-absorbing ideal of $\mathbb{Z}$. Thus the converse of Theorem 2.2 may fail. See Corollary 4.3 for the general PID case.

(b) Let R be a boolean ring. Then every proper ideal of R is an sdf-absorbing ideal of R since $x^2 = x$ for every $x \in R$.

(c) Let $R = \mathbb{Z}[W, T]$. Since $(2W)(2T) \in WTR$ and the system of linear equations $x + y = 2W$, $x - y = 2T$ has a nonzero solution in R, we have that WTR is not an sdf-absorbing ideal of R by Theorem 2.7. However, note that WTR is a radical ideal of R.

(d) Let $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}$ and $I = I_1 \times \cdots \times I_n \times J$ be an ideal of R. Using Theorem 2.7 again, one can easily verify that I is an sdf-absorbing ideal of R if and only if J is a prime ideal of $\mathbb{Z}$ or $J = 2q\mathbb{Z}$ for some odd prime integer q.

(e) Let $R = K[X]$, where K is a field, and $I = (X+1)(X-1)R$. Then I is a radical ideal of R if and only if $\mathrm{char}(K) \neq 2$. Thus I is never an sdf-absorbing ideal of R by Theorems 2.2 and 2.6.

(f) Let $R = K[X]$, where K is a field, and $I = f_1^{n_1} \cdots k_k^{n_k} R$ for nonassociate irreducible $f_1, \ldots, f_k \in R$ and positive integers $n_1, \ldots, n_k$. Then I is a radical (respectively, prime) ideal of R if and only if $n_1 = \cdots = n_k = 1$ (respectively, $k = n_k = 1$). If $\mathrm{char}(K) = 2$, then I is an sdf-absorbing ideal of R if and only if I is a radical ideal of R by Theorem 2.4. If $\mathrm{char}(K) \neq 2$, then I is an sdf-absorbing ideal of R if and only if I is a prime ideal of R by Theorem 2.6.

(g) Let R be a valuation domain. Then every radical ideal of R is a prime ideal [10, Theorem 17.1(2)]; so the prime ideals are the only sdf-absorbing ideals of R by Theorem 2.2.

The next two theorems and corollary follow directly from the definitions and Remark 2.3(b); so their proofs are omitted.

Theorem 2.9. *Let I be an sdf-absorbing ideal of a commutative ring R, and let S be a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then I_S is an sdf-absorbing ideal of R_S.*

Theorem 2.10. *Let $f : R \to T$ be a homomorphism of commutative rings.*

(a) *If J is a nonzero sdf-absorbing ideal of T, then $f^{-1}(J)$ is an sdf-absorbing ideal of R.*

(b) *If f is injective and J is an sdf-absorbing ideal of T, then $f^{-1}(J)$ is an sdf-absorbing ideal of R.*

(c) *If f is surjective and I is an sdf-absorbing ideal of R containing* $\ker(f)$, *then $f(I)$ is an sdf-absorbing ideal of T.*

Corollary 2.11. (a) *Let $R \subseteq T$ be an extension of commutative rings and J an sdf-absorbing ideal of T. Then $J \cap R$ is an sdf-absorbing ideal of R.*

(b) *Let $J \subseteq I$ be ideals of a commutative ring R. If I is an sdf-absorbing ideal of R, then I/J is an sdf-absorbing ideal of R/J.*

(c) *If $J \subsetneq I$, then I/J is an sdf-absorbing ideal of R/J if and only if I is an sdf-absorbing ideal of R.*

The following examples show that the "nonzero" hypothesis is needed in Theorem 2.10(a) and the "$\ker(f) \subseteq I$" hypothesis is needed in Theorem 2.10(c).

Example 2.12. (a) Let $f : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}_4$ be the natural epimorphism. By Remark 2.3(a), $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_4$, but $f^{-1}(\{0\}) = 4\mathbb{Z}$ is not an sdf-absorbing ideal of $\mathbb{Z}$ by Example 2.8(a). Thus the "nonzero" hypothesis is needed in Theorem 2.10(a).

(b) Let $f : \mathbb{Z}[X] \to \mathbb{Z}$ be the epimorphism given by $f(g(X)) = g(0)$. Then $I = (X+4)$ is a prime ideal, and thus an sdf-absorbing ideal, of $\mathbb{Z}[X]$, but $f((X+4)) = 4\mathbb{Z}$ is not an sdf-absorbing ideal of $\mathbb{Z}$ by Example 2.8(a). Note that

$\ker(f) = (X) \not\subseteq (X+4) = I$; so the "$\ker(f) \subseteq I$" hypothesis is needed in Theorem 2.10(c).

3. When Every Ideal is an sdf-Absorbing Ideal

In this section, we consider when every proper ideal or nonzero proper ideal of a commutative ring is an sdf-absorbing ideal. Recall that every proper ideal of a commutative ring R is a radical ideal if and only if R is von Neumann regular (cf. [1, Proposition 1.1]). We use this fact to show that if every nonzero proper ideal of R is an sdf-absorbing ideal of R, then $R/\mathrm{nil}(R)$ is von Neumann regular. In particular, if in addition R is reduced, then R is von Neumann regular.

Theorem 3.1. *Let R be commutative ring such that every nonzero proper ideal of R is an sdf-absorbing ideal of R. Then $R/\mathrm{nil}(R)$ is von Neumann regular. In particular, $\dim(R) = 0$. Moreover, if R is not reduced, then $\mathrm{nil}(R)$ is the unique minimal nonzero ideal of R.*

Proof. Every nonzero proper ideal of R is a radical ideal by Theorem 2.2. Thus every proper ideal of $R/\mathrm{nil}(R)$ is a radical ideal, and hence $R/\mathrm{nil}(R)$ is von Neumann regular. The "in particular" and "moreover" statements are clear. □

Example 3.2. (a) All proper ideals of $\mathbb{Z}_4$ are sdf-absorbing ideals, and all nonzero proper ideals (but not the zero ideal) of $\mathbb{Z}_{25}$ are sdf-absorbing ideals (cf. Theorem 4.11). Neither ring is reduced (i.e., von Neumann regular).

(b) All proper ideals of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are sdf-absorbing ideals, and all nonzero proper ideals (but not the zero ideal) of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are sdf-absorbing ideals (cf. Example 3.8). Both of these rings are von Neumann regular.

(c) However, not all nonzero proper ideals in a von Neumann regular ring need be sdf-absorbing ideals (cf. Example 3.8). For example, let $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then the ideal $I = \{0\} \times \{0\} \times \mathbb{Z}_3$ is not an sdf-absorbing ideal of R (to see this, let $a = (2,1,0), b = (1,1,0) \in R$).

The quasilocal case is easily handled.

Theorem 3.3. *Let R be a quasilocal commutative ring with maximal ideal M. Then every nonzero proper ideal of R is an sdf-absorbing ideal of R if and only if M is the unique prime ideal of R, M is principal, and $M^2 = \{0\}$.*

Proof. We may assume that R is not a field. Suppose that every nonzero proper ideal of R is an sdf-absorbing ideal of R. Then M is the unique prime ideal of R by Theorem 3.1. Moreover, M is the only nonzero proper ideal of R since every nonzero proper ideal of R is a radical ideal by Theorem 2.2. Thus M is principal and $M^2 = \{0\}$.

Conversely, if M is the unique prime ideal of R, M is principal, and $M^2 = \{0\}$, then M is the only nonzero proper ideal of R and is an sdf-absorbing ideal of R. □

The next example shows that in the above theorem, $\{0\}$ may or may not be an sdf-absorbing ideal of R.

Example 3.4. (a) Let $R = \mathbb{Z}_{p^2}$ for p prime. Then R satisfies the conditions of Theorem 3.3; so every nonzero proper ideal of R is an sdf-absorbing ideal of R. However, $\{0\}$ is an sdf-absorbing ideal of R if and only if $p = 2$ or $p = 3$ by Theorem 4.11.

(b) Let $R = K[X]/(X^2)$, where K is a field. Then R satisfies the conditions of Theorem 3.3, so every nonzero proper ideal of R is an sdf-absorbing ideal of R. However, it is easily verified that $\{0\}$ is an sdf-absorbing ideal of R if and only if $K = \mathbb{Z}_3$.

The next several results consider the case where every proper ideal or nonzero proper ideal of a commutative von Neumann regular ring is an sdf-absorbing ideal. They depend on whether 2 is zero, a unit, or a nonzero zero-divisor in R.

Theorem 3.5. *Let R be a reduced commutative ring with $2 \in U(R)$.*

(a) *Every nonzero proper ideal of R is an sdf-absorbing ideal of R if and only if R is a field or R is isomorphic to $F_1 \times F_2$ for fields F_1, F_2.*

(b) *Every proper ideal of R is an sdf-absorbing ideal of R if and only if R is a field.*

Proof. (a) If R is a field, then the claim is clear. So assume that R is isomorphic to $F_1 \times F_2$ for fields F_1, F_2. Then R has exactly two nonzero proper ideals and each is a maximal ideal of R. Thus every nonzero proper ideal of R is an sdf-absorbing ideal of R.

Conversely, assume that R is not a field and every nonzero proper ideal of R is an sdf-absorbing ideal of R. Suppose that R has at least three distinct maximal ideals, say M_1, M_2, M_3. Then $M_1 \cap M_2 \neq \{0\}$ (if $M_1 \cap M_2 = \{0\}$, then $M_1 \subseteq M_3$ or $M_2 \subseteq M_3$). Thus $M_1 \cap M_2$ is a nonzero sdf-absorbing ideal of R and $2 \in U(R)$; so $M_1 \cap M_2$ is a prime ideal of R by Theorem 2.6, a contradiction since $\dim(R) = 0$ by Theorem 3.1. Hence, R has exactly two maximal ideals, say M_1 and M_2. Then, arguing as above, $J(R) = M_1 \cap M_2 = \{0\}$; so R is isomorphic to $F_1 \times F_2$ for fields $F_1 \cong R/M_1, F_2 \cong R/M_2$ by the Chinese Remainder Theorem.

(b) By part (a) above, we need only show that $I = \{(0, 0)\}$ is not an sdf-absorbing ideal of $R = F_1 \times F_2$ when $2 \in U(R)$. Let $a = (2, -2), b = (2, 2) \in F_1 \times F_2$. Then $a^2 - b^2 = (0, 0) \in I$, but $a + b = (4, 0) \notin I$ and $a - b = (0, -4) \notin I$ since $\mathrm{char}(F_1), \mathrm{char}(F_2) \neq 2$ as $2 \in U(R)$. Thus $I = \{(0, 0)\}$ is not an sdf-absorbing ideal of $F_1 \times F_2$. □

Since every proper ideal of a commutative von Neumann regular ring is a radical ideal, Theorem 2.4 yields the following result which generalizes the fact that every ideal of a boolean ring is an sdf-absorbing ideal (Example 2.8(b)). Note that $R = \mathbb{F}_4 \times \mathbb{F}_4$ is a von Neumann regular ring with $\mathrm{char}(R) = 2$, but R is not a boolean ring.

Theorem 3.6. *Let R be a commutative von Neumann regular ring with* $\mathrm{char}(R) = 2$. *Then every proper ideal of R is an sdf-absorbing ideal of R.*

We next handle the case when 2 is a nonzero zero-divisor of R.

Theorem 3.7. *Let R be a commutative von Neumann regular ring with* $0 \neq 2 \in Z(R)$. *Then the following statements are equivalent.*

(a) *Every proper ideal of R is an sdf-absorbing ideal of R.*
(b) *Every nonzero proper ideal of R is an sdf-absorbing ideal of R.*
(c) *Exactly one maximal ideal M of R has char$(R/M) \neq 2$.*

Proof. (a) $\Rightarrow$ (b) This is clear.

(b) $\Rightarrow$ (c) First, assume that $\mathrm{char}(R/M) = 2$ for every maximal ideal M of R. Then R is isomorphic to a subring of the direct product of fields of characteristic 2; so $\mathrm{char}(R) = 2$, a contradiction. Next, assume that R has at least two maximal ideals M_1, M_2 with $\mathrm{char}(R/M_1)$, $\mathrm{char}(R/M_2) \neq 2$. Let $I = M_1 \cap M_2$. Then $I \neq \{0\}$ since otherwise R is isomorphic to the direct product of two fields, each with characteristic $\neq 2$, by the Chinese Remainder Theorem. Thus I is an sdf-absorbing ideal of R, and hence $I/I = \{0\}$ is an sdf-absorbing ideal of R/I by Corollary 2.11(b). However, R/I is isomorphic to $F_1 \times F_2$ for fields $F_1 \cong R/M_1, F_2 \cong R/M_2$, where $\mathrm{char}(F_1)$, $\mathrm{char}(F_2) \neq 2$, by the Chinese Remainder Theorem, and thus $2 \in U(R/I)$. Hence, $\{(0,0)\}$ is not an sdf-absorbing ideal of $F_1 \times F_2$ by the proof of Theorem 3.5(b), a contradiction. Thus exactly one maximal ideal M of R has $\mathrm{char}(R/M) \neq 2$.

$(c) \Rightarrow (a)$ Suppose that exactly one maximal ideal M of R has $\mathrm{char}(R/M) \neq 2$. We need to show that every proper ideal I of R is an sdf-absorbing ideal of R. Note that I is a radical ideal of R since R is von Neumann regular. Thus it suffices to show that if $a^2 - b^2 \in I$, then either $a + b$ is in every prime (maximal) ideal of R containing I or $a - b$ is in every prime (maximal) ideal of R containing I. Let N be a maximal ideal of R containing I; we consider two cases.

First, let $N = M$ be the unique maximal ideal M of R with $\mathrm{char}(R/M) \neq 2$. If I is contained in M, then as $a^2 - b^2 \in M$, we have $a + b \in M$ or $a - b \in M$. Next, let $N \neq M$; so $\mathrm{char}(R/N) = 2$. As $a^2 - b^2 \in N$, both $a + b$ and $a - b$ are in N by Theorem 2.5, and we select the sign convention to conform with the outcome of the previous case, if necessary. So given the (radical) ideal I of R, if $a^2 - b^2 \in I$, then either $a + b$ is in every prime (maximal) ideal of R containing I or $a - b$ is in every prime (maximal) ideal of R containing I. Thus $a + b \in I$ or $a - b \in I$, and hence I is an sdf-absorbing factor ideal of R. □

Together, Theorems 3.5–3.7 completely determine when every nonzero proper ideal or proper ideal of a commutative von Neumann regular ring is an sdf-absorbing ideal since every element in a commutative von Neumann regular ring is either a unit or a zero-divisor [11, Corollary 2.4]. We next apply these criteria to a commutative von Neumann regular ring which is the direct product of finitely many fields.

Example 3.8. Let R be the direct product of finitely many fields (so R is von Neumann regular).

(a) Every proper ideal of R is an sdf-absorbing ideal of R if and only if at most one of the fields has characteristic $\neq 2$.
(b) Every nonzero proper ideal of R is an sdf-absorbing ideal of R if and only if at most one of the fields has characteristic $\neq 2$ or R is the direct product of two fields.

4. Additional Results

In this section, we give several more results about sdf-absorbing ideals in special classes of commutative rings. In particular, we determine the sdf-absorbing ideals in a PID and the direct product of two commutative rings, and study sdf-absorbing ideals in polynomial rings, idealizations, amalgamation rings, and $D+M$ constructions.

A nonzero sdf-absorbing ideal of a commutative ring R is always a radical ideal of R by Theorem 2.2; the following theorem gives a case where the converse holds.

Theorem 4.1. *Let I be a proper ideal of a commutative ring R such that $I = P_1 \cap \cdots \cap P_n$ for distinct comaximal prime ideals $P_1, \ldots, P_n$ of R. Then I is an sdf-absorbing ideal of R if and only if at most one of the P_i's has* $\mathrm{char}(R/P_i) \neq 2$.

Proof. By way of contradiction, assume that I is an sdf-absorbing ideal of R, $n \geq 2$, and $\mathrm{char}(R/P_1), \mathrm{char}(R/P_2) \neq 2$. Then $I/I = \{0\}$ is an sdf-absorbing ideal of R/I by Corollary 2.11(b), and R/I is isomorphic to $T = R/P_1 \times \cdots \times R/P_n$ by the Chinese Remainder Theorem. Let $a = (1, 1, \ldots, 1), b = (-1, 1, \ldots, 1) \in T$. Then $a^2 - b^2 = (0, \ldots, 0)$, but $a + b = (0, 2, \ldots, 2) \notin \{(0, \ldots, 0)\}$ and $a - b = (2, 0, \ldots, 0) \notin \{(0, \ldots, 0)\}$. Thus $\{(0, \ldots, 0)\}$ is not an sdf-absorbing ideal of T, a contradiction. Hence at most one of the P_i's has $\mathrm{char}(R/P_i) \neq 2$.

The converse follows easily using a slight modification to the proof of $(c) \Rightarrow (a)$ in Theorem 3.7. The details are left to the reader. □

Remark 4.2. (a) Theorem 4.1 gives criteria for $\mathrm{nil}(R)$ to be an sdf-absorbing ideal in a zero-dimensional semilocal commutative ring.
(b) In the proof of Theorem 4.1, note that if $I = I_1 \cap \cdots \cap I_n$ is an sdf-absorbing ideal of R for distinct comaximal proper ideals $I_1, \ldots, I_n$ of R, then at most one of the I_i's has $\mathrm{char}(R/I_i) \neq 2$.

Using Theorem 4.1, we have the following characterization of sdf-absorbing ideals in a PID which extends Example 2.8(a) (also, cf. Example 2.8(f), Theorems 2.4 and 2.6).

Corollary 4.3. *Let R be a PID and I a nonzero proper ideal of R.*

(a) *If 2 is a nonzero nonunit of R, then I is an sdf-absorbing ideal of R if and only if I is a prime (maximal) ideal of R, i.e. $I = aR$ for $a \in R$ prime, or $I = aR$, where $a = a_1 \cdots a_n a_{n+1}$ for nonassociate primes $a_1, \ldots, a_{n+1} \in R$ such that $a_i \,|\, 2$ in R for every $1 \leq i \leq n$. In particular, if $2 \in R$ is prime, then I is an sdf-absorbing ideal of R if and only if I is a prime (maximal) ideal of R or $I = 2pR$ for $p \in R$ prime not associate to 2.*

(b) *If $2 \in U(R)$, then I is an sdf-absorbing ideal of R if and only if I is a prime (maximal) ideal of R, i.e. $I = aR$ for $a \in R$ prime.*

(c) *If $\operatorname{char}(R) = 2$, then I is an sdf-absorbing ideal of R if and only if I is a radical ideal of R, i.e. $I = a_1 \cdots a_n R$ for nonassociate primes $a_1, \ldots, a_n \in R$.*

Proof. (a) Assume that I is a nonprime (and thus nonzero) sdf-absorbing ideal of the PID R. Then I is a radical ideal of R by Theorem 2.2, and hence I is the intersection (product) of a finite number of distinct nonzero principal prime (maximal) ideals of R. Applying Theorem 4.1, we have $I = aR$, where $a = a_1 \cdots a_n a_{n+1}$ for nonassociate prime elements $a_1, \ldots, a_{n+1} \in R$ such that $a_i \,|\, 2$ (in R) for every $1 \leq i \leq n$.

The converse is clear by Theorem 4.1. The "in particular" statement is also clear.

(b) This follows from Theorem 2.6.

(c) This follows from Theorems 2.2 and 2.4. □

We have the following example of a PID R such that $2pR$ is not an sdf-absorbing ideal of R for any prime $p \in R$.

Example 4.4. Let $R = \mathbb{Z}[i]$. Then R is a PID and $2 = -i(1+i)^2$, where $1+i$ is prime and i is a unit of R. Thus 2 is not a prime element of R and $2pR$ is not a radical ideal of R for any prime $p \in R$; so $I = 2pR$ is not an sdf-absorbing ideal of R for any prime $p \in R$ by Theorem 2.2. However, by Corollary 4.3(a), a nonprime ideal I of R is an sdf-absorbing ideal of R if and only if $I = (1+i)pR$ for some prime $p \in R$ not associate to $1+i$.

We next consider when the two ideals $I[X]$ and (I, X) are sdf-absorbing ideals of $R[X]$. The following partial result for $I[X]$ is a consequence of Theorem 4.1.

Theorem 4.5. *Let I be a proper ideal of a commutative ring R such that $I = P_1 \cap \cdots \cap P_n$ for distinct comaximal prime ideals $P_1, \ldots, P_n$ of R. Then $I[X]$ is an sdf-absorbing ideal of $R[X]$ if and only if I is an sdf-absorbing ideal of R.*

Proof. If $I[X]$ is an sdf-absorbing ideal of $R[X]$, then it is easily verified that I is an sdf-absorbing ideal of R.

Conversely, assume that I is an sdf-absorbing ideal of R. Then $I[X] = P_1[X] \cap \cdots \cap P_n[X]$, where $P_1[X], \ldots, P_n[X]$ are distinct comaximal prime ideals of $R[X]$

since $P_1, \ldots, P_n$ are distinct comaximal prime ideals of R. Moreover, at most one of the P_i's has $\mathrm{char}(R/P_i) \neq 2$ by Theorem 4.1. Since $R[X]/P_i[X]$ is isomorphic to $R/P_i[X]$ for every $1 \leq i \leq n$, at most one of the $P_i[X]$'s has $\mathrm{char}(R[X]/P_i[X]) \neq 2$. Thus $I[X]$ is an sdf-absorbing ideal of $R[X]$ by Theorem 4.1. □

Combining Theorems 4.1 and 4.5, we have the following corollary.

Corollary 4.6. *Let I be a proper ideal of a commutative ring R such that $I = P_1 \cap \cdots \cap P_n$ for distinct comaximal prime ideals $P_1, \ldots, P_n$ of R. Then the following statements are equivalent.*

(a) *I is an sdf-absorbing ideal of R.*
(b) *$I[X]$ is an sdf-absorbing ideal of $R[X]$.*
(c) *At most one of the P_i's has $\mathrm{char}(R/P_i) \neq 2$.*

Remark 4.7. Note that $\{0\}$ may be an sdf-absorbing ideal in R, but not an sdf-absorbing ideal in $R[X]$ (so, in this case, $\{0\}$ is not the intersection of finitely many distinct comaximal prime ideals of R). For example, let $R = \mathbb{Z}_4$.

The sdf-absorbing ideals (I, X) in $R[X]$ are easily classified

Theorem 4.8. *Let R be a commutative ring.*

(a) *Let I be a nonzero proper ideal of R. Then (I, X) is an sdf-absorbing ideal of $R[X]$ if and only if I is an sdf-absorbing ideal of R.*
(b) *(X) is an sdf-absorbing ideal of $R[X]$ if and only if R is reduced and $\{0\}$ is an sdf-absorbing ideal of R.*

Proof. (a) If (I, X) is an sdf-absorbing ideal of $R[X]$, then I is an sdf-absorbing ideal of R by Theorem 2.10(c).

Conversely, assume that I is a nonzero sdf-absorbing ideal of R. Let $f = a + Xm(X), g = b + Xn(X) \in R[X]$ with $f^2 - g^2 \in (I, X)$. Then $a^2 - b^2 \in I$; so $a + b \in I$ or $a - b \in I$ by Remark 2.3(b). Thus $f + g \in (I, X)$ or $f - g \in (I, X)$; so (I, X) is an sdf-absorbing ideal of $R[X]$.

(b) If (X) is an sdf-absorbing ideal of $R[X]$, then $\{0\}$ is an sdf-absorbing ideal of R by Theorem 2.10(c). Moreover, (X) is a radical ideal of $R[X]$ by Theorem 2.2; so R is also reduced.

Conversely, assume that R is reduced and $\{0\}$ is an sdf-absorbing ideal of R. Let $f = a + Xm(X), g = b + Xn(X) \in R[X]$ with $f^2 - g^2 \in (X)$. Then $a^2 - b^2 = 0$. If $a, b \neq 0$, then $a + b = 0$ or $a - b = 0$ since $\{0\}$ is an sdf-absorbing ideal of R. If $a = 0$ or $b = 0$, then $a = b = 0$ since R is reduced. So in either case, $f + g \in (X)$ or $f - g \in (X)$. Thus (X) is an sdf-absorbing ideal of $R[X]$. □

The next theorem is similar to Theorem 4.1. Note that Corollary 4.3 is also a consequence of Theorem 4.9 since in Corollary 4.3, we have $P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$.

Theorem 4.9. *Let I be a proper ideal of a commutative ring R such that $I = P_1 \cap \cdots \cap P_n$ for prime ideals $P_1, \ldots, P_n$ of R and the intersection of any $n-1$ of the ideals $P_1, \ldots, P_n$ is not equal to I. Then I is an sdf-absorbing ideal of R if and only if at most one of the P_i's has* $\mathrm{char}(R/P_i) \neq 2$.

Proof. Let I be an sdf-absorbing ideal of R. By way of contradiction, assume that $n \geq 2$ and $\mathrm{char}(R/P_1), \mathrm{char}(R/P_2) \neq 2$. Then $I \subsetneq J = P_2 \cap \cdots \cap P_n$ by hypothesis; so there is a $j \in J \backslash P_1$ and $q \in P_1 \backslash J$ (otherwise $I = P_1$). Let $x = j + q$ and $y = j - q$. Then $x \neq 0$, $y \neq 0$, and $x^2 - y^2 = 4jq \in I$. Moreover, $x + y = 2j \notin P_1$ since $2 \notin P_1$, and $x - y = 2q \notin J$ since $2 \notin P_2$. Thus $x + y \notin I$ and $x - y \notin I$; so I is not an sdf-absorbing ideal of R, a contradiction. Hence at most one of the P_i's has $\mathrm{char}(R/P_i) \neq 2$.

The converse follows easily using a slight modification to the proof of $(c) \Rightarrow (a)$ in Theorem 3.7. The details are left to the reader. □

In view of Theorem 4.9, we have the following example.

Example 4.10. Let $R = \mathbb{Z}[X_1, \ldots, X_n]$ for $n \geq 2$. Then $I = (6, 2X_1, \ldots, 2X_n, X_1, \ldots, X_n) = (X_1, 3) \cap (X_2, 2) \cap \cdots \cap (X_n, 2)$ is an sdf-absorbing ideal of R by Theorem 4.9.

The next result completely determines when $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_n$.

Theorem 4.11. $\{0\}$ *is an sdf-absorbing ideal of $\mathbb{Z}_n$ if and only if $n = 4$, $n = 9$, $n = p$ is prime, or $n = 2p$ for some odd prime p.*

Proof. Assume that $\{0\}$ is an sdf-absorbing ideal of $R = \mathbb{Z}_n$. First, suppose that $n \neq 2, 4$ is an even positive integer and $n \neq 2p$ for any odd prime p. We consider two cases. For the first case, assume that $4 \mid n$ in $\mathbb{Z}$. Then the system of linear equations $X + Y = n/2$ and $X - Y = 2$ has a solution $0 \neq x, y \in R$ and $x^2 - y^2 = 0$, but $x + y \neq 0$ and $x - y \neq 0$. Thus $\{0\}$ is not an sdf-absorbing ideal of R by Theorem 2.7. For the second case, assume that $4 \nmid n$; so n has an odd prime factor q. Then the system of linear equations $X + Y = 2n/q$ and $X - Y = 2q$ has a solution $0 \neq x, y \in R$ (note that $n \neq 2p$ by assumption) and $x^2 - y^2 = 0$, but $x + y \neq 0$ and $x - y \neq 0$. Hence, $\{0\}$ is not an sdf-absorbing ideal of R by Theorem 2.7. Next, suppose that n is odd, not prime, $n \neq 3, 9$, and let q be a prime factor of n. Then the system of linear equations $X + Y = 4n/q$ and $X - Y = 2q$ has a solution $0 \neq x, y \in R$ and $x^2 - y^2 = 0$, but $x + y \neq 0$ and $x - y \neq 0$. Thus $\{0\}$ is not an sdf-absorbing ideal of R by Theorem 2.7 again. Hence, if $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_n$, then $n = 4, n = 9, n = p$ is prime, or $n = 2p$ for some odd prime p.

Conversely, if $n = p$ is prime, then $\{0\}$ is a maximal ideal, and thus an sdf-absorbing ideal, of the field $\mathbb{Z}_n$. If $n = 4$ or $n = 9$, then one can easily verify that $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_n$. Finally, assume that $n = 2p$ for some odd prime p. Since $I = 2p\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$ by Example 2.8(a), we have that $I/I = \{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}/2p\mathbb{Z} = \mathbb{Z}_{2p}$ by Corollary 2.11(b). □

We next investigate when the direct product of two ideals is an sdf-absorbing ideal. First, we consider the case when both ideals are nonzero proper ideals.

Theorem 4.12. *Let I_1, I_2 be nonzero proper ideals of the commutative rings R_1, R_2, respectively. Then the following statements are equivalent.*

(a) *$I_1 \times I_2$ is an sdf-absorbing ideal of $R_1 \times R_2$.*
(b) *I_1, I_2 are sdf-absorbing ideals of R_1, R_2, respectively, and $2 \in I_1$ or $2 \in I_2$.*

Proof. (a) $\Rightarrow$ (b) Let $I = I_1 \times I_2$ be an sdf-absorbing ideal of $R = R_1 \times R_2$. Then it is easily shown that I_1, I_2 are sdf-absorbing ideals of R_1, R_2, respectively. Next, let $a = (1,1), b = (1,-1) \in R$. Then $a^2 - b^2 = (0,0) \in I$; so $(2,0) = a + b \in I$ or $(0,2) = a - b) \in I$. Thus $2 \in I_1$ or $2 \in I_2$.

(b) $\Rightarrow$ (a) We may assume that $2 \in I_1$. Let $(0,0) \neq a = (a_1, a_2), b = (b_1, b_2) \in R$ with $a^2 - b^2 \in I$. Then $a_1^2 - b_1^2 \in I_1$, and thus $a_1 + b_1 \in I_1$ or $a_1 - b_1 \in I_1$ by Remark 2.3(b) since I_1 is a nonzero sdf-absorbing ideal of R_1. Since $2 \in I_1$, we have $a_1 + b_1, a_1 - b_1 \in I_1$ by Theorem 2.5. Also, $a_2^2 - b_2^2 \in I_2$; so $a_2 + b_2 \in I_2$ or $a_2 - b_2 \in I_2$ by Remark 2.3(b) again since I_2 is a nonzero sdf-absorbing ideal of R_2. If $a_2 + b_2 \in I_2$, then $a + b \in I$. If $a_2 - b_2 \in I_2$, then $a - b \in I$. Thus I is an sdf-absorbing ideal of R. $\square$

Now we consider the case when one of the ideals in the product is either zero or the whole ring.

Theorem 4.13. *Let I_1, I_2 be nonzero proper ideals of the commutative rings R_1, R_2, respectively.*

(a) *$\{0\} \times R_2$ is an sdf-absorbing ideal of $R_1 \times R_2$ if and only if R_1 is reduced and $\{0\}$ is an sdf-absorbing ideal of R_1. A similar result holds for $R_1 \times \{0\}$.*
(b) *$I_1 \times R_2$ is an sdf-absorbing ideal of $R_1 \times R_2$ if and only if I_1 is an sdf-absorbing ideal of R_1. A similar result holds for $R_1 \times I_2$.*
(c) *$\{0\} \times I_2$ is an sdf-absorbing ideal of $R_1 \times R_2$ if and only if $\{0\}$ is an sdf-absorbing ideal of R_1, R_1 is reduced, I_2 is an sdf-absorbing ideal of R_2, and char$(R_1) = 2$ or $2 \in I_2$. A similar result holds for $I_1 \times \{0\}$.*
(d) *$\{0\} \times \{0\}$ is an sdf-absorbing ideal of $R_1 \times R_2$ if and only if $\{0\}$ is an sdf-absorbing ideal of R_1 and R_2, R_1 and R_2 are reduced, and char$(R_1) = 2$ or char$(R_2) = 2$.*

Proof. Let $R = R_1 \times R_2$.

(a) Assume that $\{0\} \times R_2$ is an sdf-absorbing ideal of R. Then it is clear that $\{0\}$ is an sdf-absorbing ideal of R_1. We show that R_1 is reduced. Let $c \in R_1$ with $c^2 = 0$, and $a = (c,1), b = (0,1) \in R$. Then $a^2 - b^2 = (0,0) \in \{0\} \times R_2$; so $(c,2) = a + b \in \{0\} \times R_2$ or $(c,0) = a - b \in \{0\} \times R_2$. Thus $c = 0$; so R_1 is reduced.

Conversely, assume that R_1 is reduced and $\{0\}$ is an sdf-absorbing ideal of R_1. Let $(0,0) \neq a = (a_1, a_2), b = (b_1, b_2) \in R$ with $a^2 - b^2 \in \{0\} \times R_2$. Since R_1 is reduced, we have $a_1 = b_1 = 0$ or $0 \neq a_1, b_1 \in R_1$. Hence, $a_1 + b_1 = 0$ or $a_1 - b_1 = 0$; so $a + b \in \{0\} \times R_2$ or $a - b \in \{0\} \times R_2$. Thus $\{0\} \times R_2$ is an sdf-absorbing ideal of R.

(b) The proof is similar to that of Theorem 4.12.

(c) The proof is similar to part (a) above.

(d) Let $I = \{0\} \times \{0\}$ be an sdf-absorbing ideal of R. Then it is easily shown that $\{0\}$ is an sdf-absorbing ideal of R_1 and R_2. By an argument similar to that in part (a) above, we have that R_1 and R_2 are reduced. Let $a = (1, 1), b = (1, -1) \in R$. Then $a^2 - b^2 = (0, 0) \in I$; so $(2, 0) = a + b \in I$ or $(0, 2) = a - b \in I$. Thus $\mathrm{char}(R_1) = 2$ or $\mathrm{char}(R_2) = 2$.

Conversely, assume that $\{0\}$ is an sdf-absorbing ideal of R_1 and R_2, R_1 and R_2 are reduced, and $\mathrm{char}(R_1) = 2$ or $\mathrm{char}(R_2) = 2$. We may assume that $\mathrm{char}(R_2) = 2$. Let $(0,0) \neq a = (a_1, a_2), b = (b_1, b_2) \in R$ with $a^2 - b^2 = (0, 0)$. Since R_1 is reduced, we have $a_1 = b_1 = 0$ or $0 \neq a_1, b_1 \in R_1$. Since $a_1^2 - b_1^2 = 0$, we have $a_1 + b_1 = 0$ or $a_1 - b_1 = 0$. Similarly, $a_2 + b_2 = a_2 - b_2 = 0$ since $\mathrm{char}(R_2) = 2$. If $a_1 + b_1 = 0$, then $a + b = (0, 0)$. If $a_1 - b_1 = 0$, then $a - b = (0, 0)$. Thus $I = \{(0, 0)\}$ is an sdf-absorbing ideal of R. □

Remark 4.14. (a) The previous two theorems may be combined by an abuse of definition (consider the whole ring to be an sdf-absorbing radical ideal, and note that $2 \in \{0\}$ if and only if the ring has characteristic 2). Let I_1, I_2 be ideals of R_1, R_2, respectively, not both the whole ring. Then $I_1 \times I_2$ is an sdf-absorbing ideal of $R_1 \times R_2$ if and only if I_1, I_2 are sdf-absorbing radical ideals of R_1, R_2, respectively, and $\mathrm{char}(R_1) = 2$ or $\mathrm{char}(R_2) = 2$.

(b) $\{0\}$ and $\{0, 2\}$ are sdf-absorbing ideals of $\mathbb{Z}_4$, but $\{0\} \times \{0\}$, $\{0\} \times \{0, 2\}$, and $\{0\} \times \mathbb{Z}_4$ are not sdf-absorbing ideals of $\mathbb{Z}_4 \times \mathbb{Z}_4$ by Theorem 4.13 (or choose $a = (2, 1), b = (0, 1)$). Also, see Example 4.15.

In view of Example 2.8(a), Theorem 4.12, and Theorem 4.13, we have the following example.

Example 4.15. Let $R = \mathbb{Z} \times \mathbb{Z}$ and $p \in \mathbb{Z}$ a positive prime. Then a nonzero ideal I of R is an sdf-absorbing ideal of R if and only if I is a prime ideal of R (i.e. $I = \{0\} \times \mathbb{Z}$, $I = p\mathbb{Z} \times \mathbb{Z}$, $I = \mathbb{Z} \times \{0\}$, or $I = \mathbb{Z} \times p\mathbb{Z}$), $I = 2\mathbb{Z} \times p\mathbb{Z}$, $I = p\mathbb{Z} \times 2\mathbb{Z}$, $I = 2p\mathbb{Z} \times \mathbb{Z}$ $(p \neq 2)$, $I = \mathbb{Z} \times 2p\mathbb{Z}$ $(p \neq 2)$, $I = 2\mathbb{Z} \times 2p\mathbb{Z}$ $(p \neq 2)$, $I = 2p\mathbb{Z} \times 2\mathbb{Z}$ $(p \neq 2)$, $I = \{0\} \times 2\mathbb{Z}$, or $I = 2\mathbb{Z} \times \{0\}$.

The ideals $\{0\} \times \{0\}, \{0\} \times p\mathbb{Z}$ $(p \neq 2)$, $p\mathbb{Z} \times \{0\}$ $(p \neq 2)$, $\{0\} \times 2p\mathbb{Z}$ $(p \neq 2)$, and $2p\mathbb{Z} \times \{0\}$ $(p \neq 2)$ are not sdf-absorbing ideals of R (or choose $a = (1, 1)$, $b = (1, -1)$).

In the following result, we determine the sdf-absorbing ideals in idealization rings. Recall that for a commutative ring R and R-module M, the *idealization of R and M* is the commutative ring $R(+)M = R \times M$ with identity $(1,0)$ under addition defined by $(r,m)+(s,n) = (r+s, m+n)$ and multiplication defined by $(r,m)(s,n) = (rs, rn+sm)$. For more on idealizations, see [5, 11]. Every ideal of $R(+)M$ has the form $I(+)N$ for I an ideal of R and N a submodule of M [11, Theorem 25.1(1)]; so Theorem 4.16 completely determines the nonzero sdf-absorbing ideals of $R(+)M$.

Theorem 4.16. *Let R be a commutative ring, I a nonzero proper ideal of R, M an R-module, and N a submodule of M. Then $I(+)N$ is an sdf-absorbing ideal of $R(+)M$ if and only if I is an sdf-absorbing ideal of R and $N = M$.*

Proof. Let $A = R(+)M$ and $J = I(+)N$.

Assume that J is an sdf-absorbing ideal of A. It is easily verified that I is an sdf-absorbing ideal of R. By way of contradiction, assume that $N \subsetneq M$; so there is an $m \in M \backslash N$. Let $0 \neq i \in I$, and $a = (i,0), b = (0,m) \in R(+)M = A$. Then $a^2 - b^2 = (i^2, 0) \in I(+)N = J$, but $a+b = (i,m) \notin J$ and $a - b = (i,-m) \notin J$, a contradiction. Thus $N = M$.

Conversely, assume that I is an sdf-absorbing ideal of R and $N = M$. Let $a^2 - b^2 \in J$ for $(0,0) \neq a, b \in A$, where $a = (a_1, m_1)$ and $b = (b_1, m_2)$. Since I is a nonzero sdf-absorbing ideal of R and $a_1^2 - b_1^2 \in I$, we have $a_1 + b_1 \in I$ or $a_1 - b_1 \in I$ by Remark 2.3(b). If $a_1 + b_1 \in I$, then $a + b \in I(+)M = J$. If $a_1 - b_1 \in I$, then $a - b \in I(+)M = J$. Thus $J = I(+)M$ is an sdf-absorbing ideal of A. □

The following example shows that it is crucial that I be a nonzero ideal in Theorem 4.16.

Example 4.17. Let $R = \mathbb{Z}_4$, $M = N = \mathbb{Z}_4$, and $I = \{0\}$. Then $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_4$, but $\{0\}(+)\mathbb{Z}_4$ is not an sdf-absorbing ideal of $\mathbb{Z}_4(+)\mathbb{Z}_4$ by Theorem 2.2 since $\{0\}(+)\mathbb{Z}_4$ is not a radical ideal of $\mathbb{Z}_4(+)\mathbb{Z}_4$ (or consider $x = (2,0)$, $y = (0,2)$).

Next, we consider when $\{0\}(+)N$ is an sdf-absorbing ideal of $R(+)M$.

Remark 4.18. Let R be a commutative ring and M a nonzero R-module.

(a) If $\{0\}(+)N$ is an sdf-absorbing ideal of $R(+)M$ for N a proper submodule of M, then $N = \{0\}$ by Theorem 2.2.
(b) It is easily shown that $\{0\}(+)M$ is an sdf-absorbing ideal of $R(+)M$ if and only if R is reduced and $\{0\}$ is an sdf-absorbing ideal of R (cf. Example 4.17).
(c) It is easily shown that $\{(0,0)\}$ is not an sdf-absorbing ideal of $R(+)M$ when $|M| \neq 3$. However, $\{(0,0)\}$ is an sdf-absorbing ideal of $\mathbb{Z}_3(+)\mathbb{Z}_3$, but not of $\mathbb{Z}(+)\mathbb{Z}_3$.

In the next result, we study sdf-absorbing ideals in amalgamation rings. Let A, B be commutative rings, $f : A \to B$ a homomorphism, and J an ideal of B. Recall that the *amalgamation of A and B with respect to f along J* is the subring $A \bowtie_J B = \{(a, f(a) + j) \mid a \in A, j \in J\}$ of $A \times B$.

Theorem 4.19. *Let A and B be commutative rings, $f : A \to B$ a homomorphism, J an ideal of B, and I a nonzero proper ideal of A. Then $I \bowtie_J B$ is an sdf-absorbing ideal of $A \bowtie_J B$ if and only if I is an sdf-absorbing ideal of A.*

Proof. If $I \bowtie_J B$ is an sdf-absorbing ideal of $A \bowtie_J B$, then it is easily verified I is an sdf-absorbing ideal of A.

Conversely, assume that I is a nonzero sdf-absorbing ideal of A. Let $x = (a, f(a) + j_1), y = (b, f(b) + j_2) \in A \bowtie_J B$ such that $x^2 - y^2 \in I \bowtie_J B$. Since $a^2 - b^2 \in I$ and I is a nonzero sdf-absorbing ideal of A, we have $a + b \in I$ or $a - b \in I$ by Remark 2.3(b). If $a + b \in I$, then $x + y = (a + b, f(a) + j_1 + f(b) + j_2) = (a + b, f(a + b) + j_1 + j_2) \in I \bowtie_J B$. Similarly, if $a - b \in I$, then $x - y = (a - b, f(a - b) + j_1 - j_2) \in I \bowtie_J B$. Thus $I \bowtie_J B$ is an sdf-absorbing ideal of $A \bowtie_J B$. $\square$

The following example shows that it is again crucial that I be a nonzero ideal in Theorem 4.19.

Example 4.20. $A = B = J = \mathbb{Z}_4$, $f = 1_A : A \to A$, and $I = \{0\}$. Then $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_4$, but $\{0\} \bowtie_{\mathbb{Z}_4} \mathbb{Z}_4$ is not an sdf-absorbing ideal of $\mathbb{Z}_4 \bowtie_{\mathbb{Z}_4} \mathbb{Z}_4$ by Theorem 2.2 since $\{0\} \bowtie_{\mathbb{Z}_4} \mathbb{Z}_4 \neq \{(0, 0)\}$ is not a radical ideal of $\mathbb{Z}_4 \bowtie_{\mathbb{Z}_4} \mathbb{Z}_4$ (or consider $x = (2, 0), y = (0, 2)$).

Let T be an integral domain of the form $K + M$, where the field K is a subring of T and M is a nonzero maximal ideal of T, and let D be a subring of K. Then $R = D + M$ is a subring of T with the same quotient field as T. This "$D + M$" construction has proved very useful for constructing examples since ring-theoretic properties of R are often determined by those of T and D. The "classical" case, when T is a valuation domain, was first studied systemically in [9, Appendix II], and the "generalized" $D + M$ construction as above was introduced and studied in [8]. The next several results concern the sdf-absorbing ideals in $D + M$. (The relevant facts concerning ideals in $D + M$ used in the proof of Theorem 4.21 may be found in [9, Theorem A, p. 560] or [10, Exercise 11, p. 202].)

Theorem 4.21. *Let $T = K + M$ be an integral domain, where the field K is a subring of T and M is a nonzero maximal ideal of T, and let D be a subring of K and $R = D + M$.*

(a) *Let I be an ideal of D. Then $I + M$ is an sdf-absorbing ideal of R if and only if I is an sdf-absorbing ideal of D.*

(b) *Let T be a valuation domain. Then J is an sdf-absorbing ideal of R if and only if $J = I + M$, where I is an sdf-absorbing ideal of D, or J is a prime ideal of T.*

Proof. (a) This follows directly from Theorem 2.10(a)–(c).

(b) Let J be an ideal of R; so J is comparable to M. If $M \subseteq J$, then $J = I + M$ for an ideal I of D. Thus J is an sdf-absorbing ideal of R if and only if I is an sdf-absorbing ideal of D by part (a) above. So we may assume that $J \subseteq M$. Note that the prime ideals of R contained in M are just the prime ideals of T. Hence, the radical ideals of R contained in M are precisely the prime ideals of T (since radical ideals in a valuation domain are prime). Thus J is an sdf-absorbing ideal of R if and only if J is a prime ideal of T. □

Example 4.22. Let $T = \mathbb{Q}[[X]] = \mathbb{Q} + X\mathbb{Q}[[X]]$, $R = \mathbb{Z} + X\mathbb{Q}[[X]]$, and $p \in \mathbb{Z}$ a positive prime. Then T is a valuation domain (DVR) with maximal ideal $M = X\mathbb{Q}[[X]]$. Thus the sdf-absorbing ideals of R are the prime ideals $\{0\}$, $X\mathbb{Q}[[X]]$, and $p\mathbb{Z} + X\mathbb{Q}[[X]]$, and the ideals $2p\mathbb{Z} + X\mathbb{Q}[[X]]$ $(p \neq 2)$ by Theorem 4.21(b) and Example 2.8(a). Note that R is a Bézout domain by [8, Theorem 7], but not a PID since M is not a principal (or even finitely generated) ideal of R.

5. Weakly Square-Difference Factor Absorbing Ideals

Recall from [4] (also see [7]) that a proper ideal I of a commutative ring R is a *weakly prime ideal* of R if whenever $0 \neq ab \in I$ for $a, b \in R$, then $a \in I$ or $b \in I$. In this section, we introduce and study the "weakly" analog of sdf-absorbing ideals. First we give the definition.

Definition 5.1. A proper ideal I of a commutative ring R is a *weakly square-difference factor absorbing ideal* (weakly sdf-absorbing ideal) of R if whenever $0 \neq a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b \in I$ or $a - b \in I$.

A weakly prime ideal or sdf-absorbing ideal of R is clearly also a weakly sdf-absorbing ideal of R. If R is an integral domain, then I is a weakly prime (respectively, weakly sdf-absorbing) ideal of R if and only if it is a prime (respectively, sdf-absorbing) ideal of R. Also, $\{0\}$ is vacuously a weakly prime and weakly sdf-absorbing ideal of R, but need not be a prime or sdf-absorbing ideal of R. The following is an example of a nonzero weakly sdf-absorbing ideal that is neither an sdf-absorbing ideal nor a weakly prime ideal.

Example 5.2. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$, and $I = \{0\} \times \{0, 2\}$. Then I is not a radical ideal of R; so I is not an sdf-absorbing ideal of R by Theorem 2.2. Also, $(0, 0) \neq (2, 2)(0, 1) \in I$, but $(2, 2) \notin I$ and $(0, 1) \notin I$; so I is not a weakly prime ideal of R. Note that if $x^2 - y^2 \in I$ for $x, y \in R$, then $x^2 - y^2 = (0, 0)$. Thus I is a weakly sdf-absorbing ideal of R.

The next theorem is the "weakly" version of Theorem 2.6.

Theorem 5.3. *Let I be a weakly sdf-absorbing ideal of a commutative ring R with $2 \in U(R)$. Then I is a weakly prime ideal of R.*

Proof. The proof is similar to the proof of Theorem 2.6. The details are left to the reader. $\square$

The next two theorems and corollary are the "weakly" analogs of Theorem 2.9, Theorem 2.10(b) and (c), and Corollary 2.11(a) and (b), respectively. They follow directly from the definitions; so their proofs are omitted.

Theorem 5.4. *Let I be a weakly sdf-absorbing ideal of a commutative ring R, and let S be a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then I_S is a weakly sdf-absorbing ideal of R_S.*

Theorem 5.5. *Let $f : R \to T$ be a homomorphism of commutative rings.*

(a) *If f is injective and J is a weakly sdf-absorbing ideal of T, then $f^{-1}(J)$ is a weakly sdf-absorbing ideal of R.*
(b) *If f is surjective and I is a weakly sdf-absorbing ideal of R containing* $\ker(f)$, *then $f(I)$ is a weakly sdf-absorbing ideal of T.*

Corollary 5.6. (a) *Let $R \subseteq T$ be an extension of commutative rings and J a weakly sdf-absorbing ideal of T. Then $J \cap R$ is a weakly sdf-absorbing ideal of R.*
(b) *Let $J \subseteq I$ be ideals of a commutative ring R. If I is a weakly sdf-absorbing ideal of R, then I/J is a weakly sdf-absorbing ideal of R/J.*

The following examples (cf. Example 2.12) show that the "weakly" analog of Theorem 2.10(a) and Corollary 2.11(c) may fail and the "$\ker(f) \subseteq I$" hypothesis is needed in Theorem 5.5(b).

Example 5.7. (a) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}_4 \times \mathbb{Z}_4$ be the natural epimorphism. By Example 5.2, $\{0\} \times \{0, 2\}$ is a weakly sdf-absorbing ideal of $\mathbb{Z}_4 \times \mathbb{Z}_4$, but $f^{-1}(\{0\} \times \{0, 2\}) = 4\mathbb{Z} \times 2\mathbb{Z}$ is not a weakly sdf-absorbing ideal of $\mathbb{Z} \times \mathbb{Z}$ (let $a = (2, 2)$ and $b = (0, 2)$). Thus the "weakly" analog of Theorem 2.10(a) and Corollary 2.11(c) may fail.
(b) Let $f : \mathbb{Z}[X] \to \mathbb{Z}$ be the epimorphism given by $f(g(X)) = g(0)$. Then $I = (X + 4)$ is a prime ideal, and thus a weakly sdf-absorbing ideal, of $\mathbb{Z}[X]$, but $f((X + 4)) = 4\mathbb{Z}$ is not a weakly sdf-absorbing ideal of $\mathbb{Z}$ (let $a = 4$ and $b = 2$). Note that $\ker(f) = (X) \not\subseteq (X + 4) = I$; so the "$\ker(f) \subseteq I$" hypothesis is needed in Theorem 5.5(b).

If I is a weakly prime ideal of a commutative ring R that is not a prime ideal, then $I \subseteq \mathrm{nil}(R)$ by [4, Theorem 1]. A similar result holds for weakly sdf-absorbing ideals.

Theorem 5.8. *Let I be a weakly sdf-absorbing ideal of a commutative ring R. If I is not an sdf-absorbing ideal of R, then $I \subseteq \mathrm{nil}(R)$.*

Proof. Since I is not an sdf-absorbing ideal of R, we have $a^2 - b^2 = 0$ for some $0 \neq a, b \in R$, but $a + b \notin I$ and $a - b \notin I$. Note that if $a, b \in I$, then $a + b \in I$ and $a - b \in I$, a contradiction. So without loss of generality, we may assume that $b \notin I$. Let $i \in I$. Then $b + i, b - i \neq 0$. We first show that $a^2 - (b + i)^2 = 0$ and $a^2 - (b - i)^2 = 0$. Since $i \in I$ and $a^2 - b^2 = 0$, we have $a^2 - (b + i)^2 = a^2 - b^2 - 2bi - i^2 = -2bi - i^2 \in I$. Suppose that $a^2 - (b + i)^2 \neq 0$. Since I is a weakly sdf-absorbing ideal of R, either $a + (b + i) \in I$ or $a - (b + i) \in I$. Thus $a + b \in I$ or $a - b \in I$, a contradiction. Similarly, $a^2 - (b - i)^2 = 0$. Hence, $-2bi - i^2 = a^2 - b^2 - 2bi - i^2 = a^2 - (b + i)^2 = 0$ and $2bi - i^2 = a^2 - b^2 + 2bi - i^2 = a^2 - (b - i)^2 = 0$; so $2i^2 = 0$. Thus $2i \in \mathrm{nil}(R)$. Since $2bi - i^2 = 0$ and $2i \in \mathrm{nil}(R)$, we have $i^2 = 2bi \in \mathrm{nil}(R)$. Hence $i \in \mathrm{nil}(R)$, and thus $I \subseteq \mathrm{nil}(R)$. □

In light of the proof of Theorem 5.8, we have the following result.

Corollary 5.9. *Let I be a weakly sdf-absorbing ideal of a commutative ring R that is not an sdf-absorbing ideal of R.*

(a) $2i^2 = 0$, *and hence* $2i \in \mathrm{nil}(R)$, *for every* $i \in I$. *Moreover, if* $2 \notin Z(R)$ *or* *char*$(R) = 2$, *then* $i^2 = 0$ *for every* $i \in I$.
(b) *If R is reduced, then $I = \{0\}$.*

We next investigate when $I \times J$ is a weakly sdf-absorbing ideal of $R_1 \times R_2$.

Theorem 5.10. *Let R_1, R_2 be commutative rings and I a nonzero weakly sdf-absorbing ideal of R_1. Then the following statements are equivalent.*

(a) *$I \times R_2$ is a weakly sdf-absorbing ideal of $R_1 \times R_2$.*
(b) *I is an sdf-absorbing ideal of R_1.*
(c) *$I \times R_2$ is an sdf-absorbing ideal of $R_1 \times R_2$.*

Proof. (a) $\Rightarrow$ (b) Let $R = R_1 \times R_2$ and $J = I \times R_2$. Assume by way of contradiction that I is not an sdf-absorbing ideal of R_1. Since I is a weakly sdf-absorbing ideal of R_1, there are $0 \neq a, b \in R_1$ such that $a^2 - b^2 = 0$, but $a + b \notin I$ and $a - b \notin I$. Let $x = (a, 1), y = (b, 0) \in R$. Then $0 \neq x, y \in R$ and $(0, 0) \neq x^2 - y^2 \in J$. Since J is a weakly sdf-absorbing ideal of R, we have $x + y = (a + b, 1) \in J$ or $x - y = (a - b, 1) \in J$. Thus $a + b \in I$ or $a - b \in I$, a contradiction. Hence I is an sdf-absorbing ideal of R_1.

(b) $\Rightarrow$ (c) This follows from Remark 4.14(b).

(c) $\Rightarrow$ (a) This is clear. □

Theorem 5.11. *Let R_1, R_2 be commutative rings, I a weakly sdf-absorbing ideal of R_1 that is not an sdf-absorbing ideal, and J a weakly sdf-absorbing ideal of R_2*

that is not an sdf-absorbing ideal. Then the following statements are equivalent

(a) *$I \times J$ is a weakly sdf-absorbing ideal of $R_1 \times R_2$ that is not an sdf-absorbing ideal.*
(b) *$I \times J$ is a weakly sdf-absorbing ideal of $R_1 \times R_2$.*
(c) *If $a^2 - b^2 \in I$ for $a, b \in R_1$, then $a^2 - b^2 = 0$, and if $c^2 - d^2 \in J$ for $c, d \in R_2$, then $c^2 - d^2 = 0$.*
(d) *If $x^2 - y^2 \in I \times J$ for $0 \neq x, y \in R_1 \times R_2$, then $x^2 - y^2 = (0, 0)$.*

Proof. Let $R = R_1 \times R_2$ and $K = I \times J$.

(a) $\Rightarrow$ (b) This is clear.

(b) $\Rightarrow$ (c) Assume $0 \neq a^2 - b^2 \in I$ for $a, b \in R_1$. Since J is a weakly sdf-absorbing ideal of R_2 that is not an sdf-absorbing ideal, we have $e^2 - f^2 = 0$ for some $0 \neq e, f \in R_2$, but $e + f \notin J$ and $e - f \notin J$. Let $x = (a, e)$ and $y = (b, f)$. Then $0 \neq x, y \in R$ and $0 \neq x^2 - y^2 = (a^2 - b^2, e^2 - f^2) \in K$. Since K is a weakly sdf-absorbing ideal of R, we have $x + y = (a + b, e + f) \in K$ or $x - y = (a - b, e - f) \in K$. Thus $e + f \in J$ or $e - f \in J$, a contradiction. Hence, if $a^2 - b^2 \in I$ for $a, b \in R_1$, then $a^2 - b^2 = 0$. A similar argument shows that if $c^2 - d^2 \in J$ for $c, d \in R_2$, then $c^2 - d^2 = 0$.

(c) $\Rightarrow$ (d) This is clear.

(d) $\Rightarrow$ (a) Clearly K is a weakly sdf-absorbing ideal of R. We show that K is not an sdf-absorbing ideal of R. Since I is a weakly sdf-absorbing ideal of R_1 that is not an sdf-absorbing ideal, we have $a^2 - b^2 = 0$ for some $0 \neq a, b \in R_1$, but $a + b \notin I$ and $a - b \notin I$. Let $x = (a, 0), y = (0, b)$. Then $0 \neq x, y \in R$ and $x^2 - y^2 = (0, 0) \in K$, but $x + y = (a + b, 0) \notin K$ and $x - y = (a - b, 0) \notin K$. Hence, K is not an sdf-absorbing ideal of R. $\square$

Note that in the proof of $(d) \Rightarrow (a)$ of Theorem 5.11 above, we only need one of I, J to not be an sdf-absorbing ideal. In view of Theorem 5.11, the following example shows that $I \times J$ may be a weakly sdf-absorbing ideal of $R_1 \times R_2$ that is not an sdf-absorbing ideal, but neither I nor J need be a weakly sdf-absorbing ideal that is not an sdf-absorbing ideal.

Example 5.12. Let $R_1 = R_2 = \mathbb{Z}_4$, $R = R_1 \times R_2$, and $K = \{0\} \times \{0, 2\}$. Then K is a nonzero weakly sdf-absorbing ideal of R that is not an sdf-absorbing ideal by Example 5.2. However, $I = \{0\}$ is an sdf-absorbing ideal of R_1 and $J = \{0, 2\}$ is an sdf-absorbing ideal of R_2.

The following example satisfies the hypothesis of Theorem 5.11.

Example 5.13. Let $R_1 = \mathbb{Z}_2[X]/(X^2)$, $R_2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, and $R = R_1 \times R_2$. Then $I = \{0\}$ is a weakly sdf-absorbing ideal of R_1. Since $(X + 1)^2 - 1^2 = 0$ in R_1, but $X \notin I$, we have that I is not an sdf-absorbing ideal of R_1. Let $J = \{0\} \times \{0, 2\}$.

Then J is a weakly sdf-absorbing ideal of R_2 that is not an sdf-absorbing ideal by Example 5.2. Since $x^2 - y^2 \in K = I \times J$ for $0 \neq x, y \in R$ implies $x^2 - y^2 = (0, 0, 0) \in R$, we have that $K = I \times J$ is a weakly sdf-absorbing ideal of R that is not an sdf-absorbing ideal by Theorem 5.11.

In Theorem 4.11, we determined when $\{0\}$ is an sdf-absorbing ideal of $\mathbb{Z}_n$. We next consider $\mathrm{nil}(\mathbb{Z}_n)$ $(= J(\mathbb{Z}_n))$.

Theorem 5.14. (a) $\mathrm{nil}(\mathbb{Z}_n)$ *is an sdf-absorbing ideal of* $\mathbb{Z}_n$ *if and only if* $n = q^m$ *for some integer* $m \geq 1$ *and positive prime* q *or* $n = 2^i p^k$ *for some integers* $i, k \geq 1$ *and positive prime* $p \neq 2$.

(b) $\mathrm{nil}(\mathbb{Z}_n) = \{0\}$ *is a weakly sdf-absorbing ideal of* $\mathbb{Z}_n$ *that is not an sdf-absorbing ideal of* $\mathbb{Z}_n$ *if and only if* $n = pq$ *for distinct odd positive primes* p, q *or* $n = p_1, \ldots, p_m$ *for distinct positive primes* $p_1, \ldots, p_m$, *where* $m \geq 3$.

Proof. (a) This follows from Theorem 4.9.

(b) Note that $\mathrm{nil}(\mathbb{Z}_n) = \{0\}$ if and only if n is a product of distinct positive primes. The result then follows from Theorems 4.9 and 4.11. □

ORCID

Ayman Badawi https://orcid.org/0000-0003-1257-5955

Jim Coykendall https://orcid.org/0000-0003-4893-5852

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